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COMMENT

Reply to Comment on 'Dirac theory in spacetime algebra'

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Abstract

The Dirac theory formulated by Joyce (Joyce W P 2001 *J. Phys. A: Math. Gen.* **34** 1991–2005) is equivalent to two copies of the usual Dirac formulation. The comment of Baylis (Baylis W E 2002 *J. Phys. A: Math. Gen.* **35** 4791) concerns the extension of the Joyce formulation to the entire Dirac algebra $\mathbb{C} \otimes C\ell(3, 1)$. We demonstrate how this extended version is equivalent to four copies of the usual Dirac formulation.

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This reply discusses a comment by Baylis [1] on a paper of Joyce [2]. We adopt the notation and conventions used by Baylis. Baylis argues that the embedded Dirac theory formulation of Joyce [2], given the embedding $\mathbb{C} \otimes C\ell^+(3, 1) \subset \mathbb{C} \otimes C\ell(3, 1)$, decomposes into a usual Dirac equation and a Dirac equation with the mass reversed. The Joyce formulation is over the subalgebra $\mathbb{C} \otimes C\ell^+(3, 1)$, where the decomposition of Baylis is inapplicable.

The Joyce formulation decomposes into two copies of the usual Dirac equation (as formulated in Lounesto [3]), as we now demonstrate. We may decompose the Joyce spinor as $\psi_J = \psi_J P_{+12} + \psi_J P_{-12}$. Thus the maps $\psi \mapsto \psi P_{\pm 12}$ project onto the summands $\mathbb{C} \otimes C\ell^+(3, 1)P_{\pm 12}$. The two component Joyce formulations over $\mathbb{C} \otimes C\ell^+(3, 1)P_{+12}$ and $\mathbb{C} \otimes C\ell^+(3, 1)P_{-12}$ are equivalent. The equivalence is given by the anti-involutive map $\psi \mapsto \psi\gamma_{23}$. Each copy is equivalent to the usual Dirac equation. The invertible map for the first copy is $\psi \mapsto \psi P_{+0}$, and for the second copy $\psi \mapsto \psi P_{+0}\gamma_{23}$.

The Joyce equation over $\mathbb{C} \otimes C\ell(3, 1)$ may also be identified with four copies of the usual Dirac equation. An equivalence is valid provided it commutes with all Lorentz transformations, observables for spin and four-momentum and preserves the Dirac current. This admits the equivalence map $\psi \mapsto \gamma_{0123}\psi$. This map changes the sign of the mass on the usual, Joyce and Hestenes versions of the Dirac equation. These observations are part of a general analysis on the equivalence of Dirac formulations by Joyce and Martin [4]. In particular the usual Dirac equation over $\mathbb{C} \otimes C\ell(3, 1)P_{+0}$ is equivalent to the usual Dirac equation with the sign of the mass reversed over $\mathbb{C} \otimes C\ell(3, 1)P_{-0}$. Their equivalence is demonstrated by the map $\psi \mapsto \gamma_{0123}\psi\gamma_{01}$.

One should note that the left action of the pseudo-scalar anti-commutes with the parity (P) and time-reversal (T) operations, and these operations are transformed to $-\gamma_{0123} P \gamma_{0123}$ and $-\gamma_{0123} T \gamma_{0123}$ respectively.

Finally we remark that if two Dirac formulations are equivalent, such as among the usual Dirac, Joyce restricted to $\mathbb{C} \otimes C\ell^{+}(1, 3) P_{\pm 12}$ and Hestenes formulations, then the same physical content is realized. However, alternative mathematical procedures may be required to extract the physical information. The choice of equation is largely a matter of taste, unless one invokes extra criteria. For example, Hestenes [5] argues that the unit imaginary in the complexified spacetime algebra has no geometrical meaning and should be avoided by remaining in the real component $C\ell(1, 3)$.

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